Statistical Theory and Modeling (ST2601)
Continuous random variables

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Overview

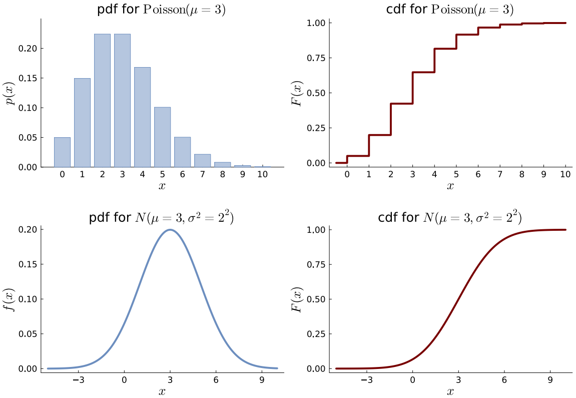
- Continuous random variables
- Exponential distribution
- Gamma distribution
- Chi2 distribution
- Beta distribution

Cumulative distribution function

- **Cumulative distribution function** (**cdf**) for a random variable X is

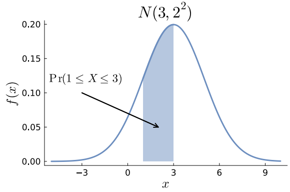
$$F(x) = \Pr(X \leq x) \qquad \text{for } -\infty < x < \infty$$

- Applies to both discrete and continuous random variables.
- The p-functions in R, for example `ppois(4, lambda = 2)`



Probability density function

- The outcome of a **continuous random variable** can be any real number, but $\Pr(X = x) = 0$ for all x! 🤯
- A **probability density function** (**pdf**) for random variable X satisfies
  - $f(x) \geq 0$ for all x, $-\infty < x < \infty$
  - $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$
  - $\Pr(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x$
- The d-functions in R. `dnorm(-1, mu = 2, sd = 1)`.



Probability density function

- The **pdf is the derivative of the cdf**:

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x)$$

- The **cdf is the integral of the pdf**:

$$F(x) = \int_{-\infty}^{x} f(t) dt$$

- **Fundamental theorem of calculus** applied to pdf-cdf pair:

$$\Pr(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x = \underbrace{F(b)}_{\Pr(X \leq b)} - \underbrace{F(a)}_{\Pr(X \leq a)}$$

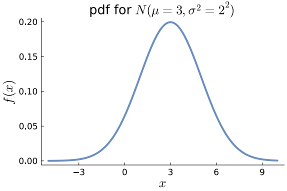
Examples of pdfs

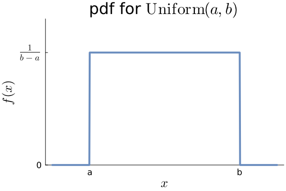
- The pdf of a **Normal** variable $X \sim N(\mu, \sigma^2)$ variable

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

- The pdf of a $X \sim \mathrm{Uniform}(a, b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$





Example pdf and cdf

- Let X be a random variable with probability density function

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- **Probability by integrating the pdf**

$$\Pr(X \leq 0.5) = \int_{-\infty}^{0.5} f(x)dx = \int_{0}^{0.5} 3x^2 dx = \left[x^3\right]_0^{0.5} = 0.5^3 = 0.125$$

- The **cdf** is

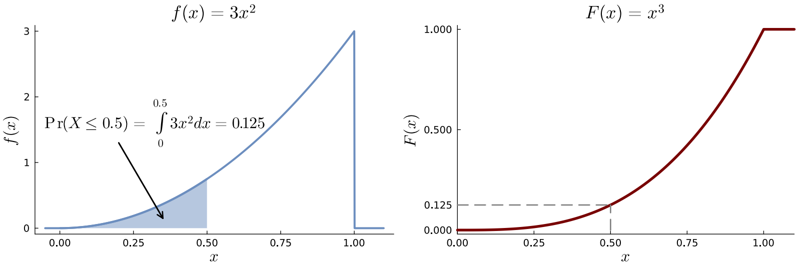
$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} 3t^2 dt = \left[t^3\right]_0^{x} = x^3$$

(note: the anti-derivative is $x^3 + C$ for some C, but here the condition $F(\infty) = F(1) = 1$ implies that $C = 0$).

- Check:

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = \frac{\mathrm{d}}{\mathrm{d}x} x^3 = 3x^2 = f(x) \qquad \texttt{OK!}$$

Example pdf and cdf



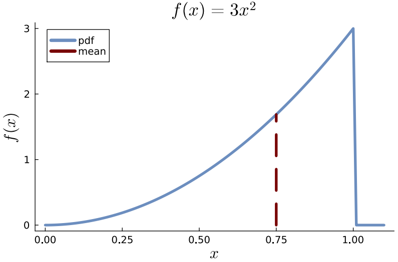
Expected value for continuous random variables

- **Expected value**

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

- Example: $f(x) = 3x^2$

$$\mu = \mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot 3x^2 dx = \int_0^1 3x^3 dx = \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}$$



Median for continuous random variables

- **Median** $\mathrm{med}(X)$ for a random variable X is the smallest x that satisfies:

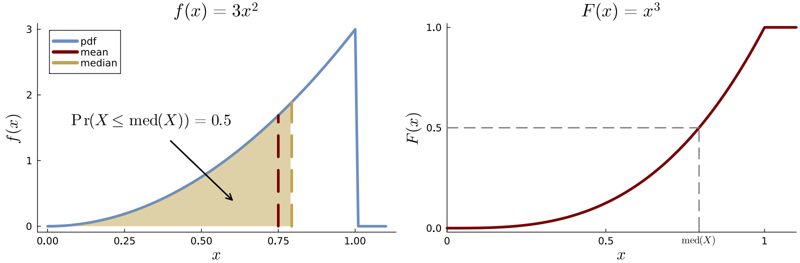
$$F(x) = 0.5$$

- Discrete variables: $\mathrm{med}(X)$ is the smallest x that satisfies:

$$F(x) \geq 0.5$$

- Example: $f(x) = 3x^2$

$$F(x) = x^3 = 0.5 \quad \Longrightarrow \quad x = 0.5^{1/3} \approx 0.794$$



Variance for continuous random variables

- **Variance**

$$\mathbb{V}(X) = \mathbb{E}\left((X-\mu)^2\right) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx$$

or

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mu^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot 3x^2 dx = \int_{-\infty}^{\infty} 3x^4 dx = \left[\frac{3}{5}x^5\right]_0^1 = \frac{3}{5}$$

So

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mu^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = 0.0375$$

- The standard deviation is

$$\mathbb{S}(X) = \sqrt{\mathbb{V}(X)} = \sqrt{0.0375} \approx 0.194.$$

Exponential distribution

- $X \sim \mathrm{Expon}(\beta)$ with support $X \in [0, \infty)$.
- Parameters: $\beta > 0$.
- Data as life time, duration etc. Memoryless.
- Probability density function

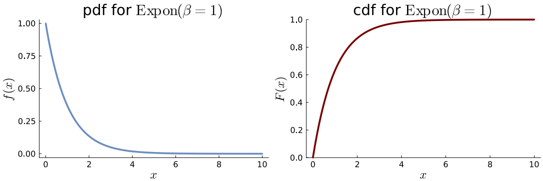
$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}} \quad \text{for } x \geq 0$$

- Cumulative distribution function

$$F(x) = \int_0^x \frac{1}{\beta} e^{-\frac{t}{\beta}} dt = \left[-e^{-\frac{t}{\beta}} \right]_0^x = -e^{-\frac{x}{\beta}} - (-1) = 1 - e^{-\frac{x}{\beta}}$$

- **Mean** and **variance**

$$\mathbb{E}(X) = \beta \qquad \mathbb{V}(X) = \beta^2$$



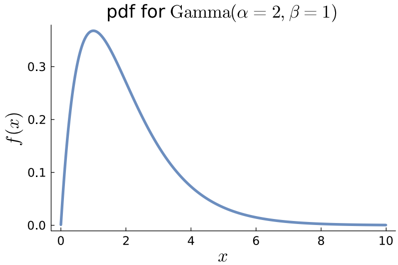
Gamma distribution

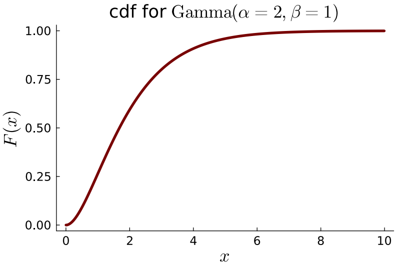
- $X \sim \mathrm{Gamma}(\alpha, \beta)$ with support $X \in [0, \infty)$.
- Parameters: $\alpha > 0$, $\beta > 0$.
- Probability density function (**scale** parameteriztion)

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad \text{for } x \geq 0$$

- $\mathrm{Expon}(\beta)$ is the special case $\mathrm{Gamma}(\alpha = 1, \beta)$.
- **Mean** and **variance**

$$\mathbb{E}(X) = \alpha\beta \qquad \mathbb{V}(X) = \alpha\beta^2$$





The Gamma function

- **Gamma function**

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

- Fundamental property

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

- When x is an integer

$$\Gamma(x) = (x-1)!$$

- `gamma(x)` in R.

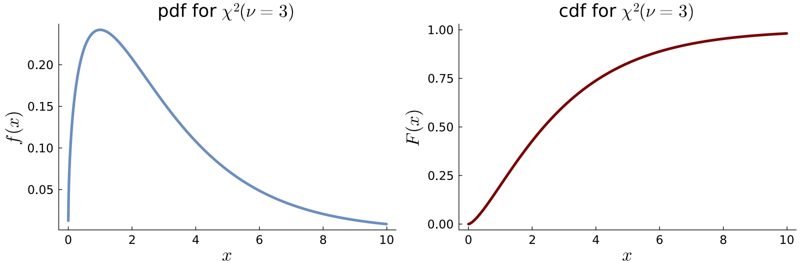
Chi-squared distribution

- The special case $\mathrm{Gamma}(\alpha = \nu/2, \beta = 2)$ is the **Chi-squared (χ^2) distribution** with ν degrees of freedom.
- We write $X \sim \chi^2_\nu$.
- Important distribution only because of

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(0,1) \quad \text{then} \quad \sum_{i=1}^{n} X_i^2 \sim \chi^2_n$$

- **Mean** and **variance**

$$\mathbb{E}(X) = \nu \qquad \mathbb{V}(X) = 2\nu$$

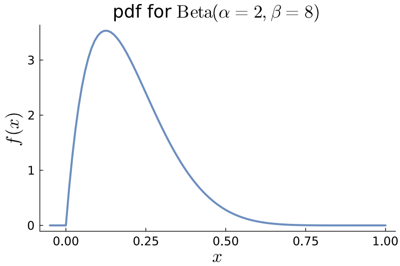


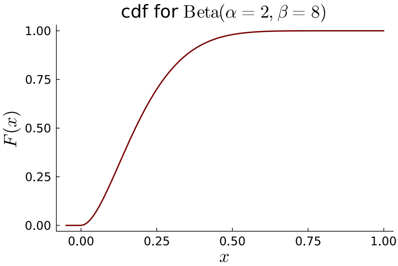
Beta distribution

- $X \sim \mathrm{Beta}(\alpha, \beta)$ with support $X \in (0, 1)$.
- Parameters: $\alpha > 0$, $\beta > 0$.
- Data as **proportions**:
  - $X = \frac{\text{firm own capital}}{\text{firm total capital}}$
  - $X = \%$bleached coral.
- Probability density function

$$f(x) = \frac{1}{\mathrm{B}(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad \text{for } 0 < x < 1$$

- $\mathrm{B}(\alpha, \beta)$ is the **Beta function**. `beta(a, b)` in R.





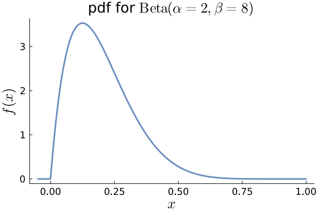
Beta distribution

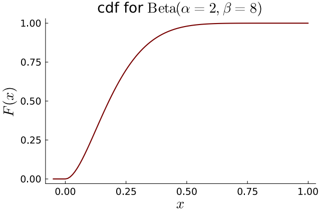
- $X \sim \mathrm{Beta}(\alpha, \beta)$ with support $X \in (0, 1)$.
- Parameters: $\alpha > 0$, $\beta > 0$.
- Data as **proportions**: $X = \frac{\text{own capital}}{\text{total capital}}$ or $X = \%$bleached coral.
- Probability density function

$$f(x) = \frac{1}{\mathrm{B}(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad \text{for } 0 < x < 1$$

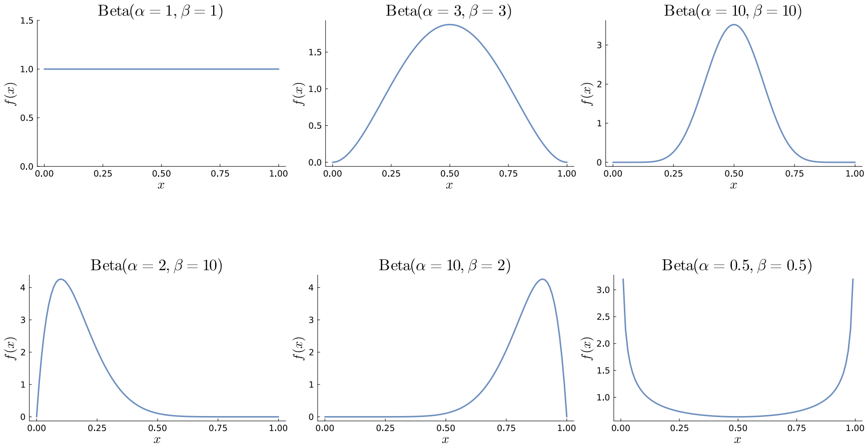
where $B(\alpha, \beta)$ is the **Beta function** (`beta(a, b)` in R):

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$





Beta distribution



Beta distribution

- $X \sim \mathrm{Beta}(\alpha, \beta)$ then

$$\mathbb{E}(X) = \frac{\alpha}{\alpha + \beta}$$
$$\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

- When $\alpha = \beta$, the Beta distribution is symmetric around the mean.

- Larger α and β gives a more concentrated distribution (smaller variance).

Inference - parameter estimation

- Probability distributions have parameters:
  - Exponential β
  - Normal μ, σ^2
  - Beta α, β
- We learn (**estimate**) such parameters from data.
- Example: $X =$ proportion of crude oil converted to gasoline.
- **Fitting** a $\mathrm{Beta}(\alpha, \beta)$ distribution using **maximum likelihood**.
- Estimates: $\hat{\alpha} = 2.504$ and $\hat{\beta} = 10.233$.